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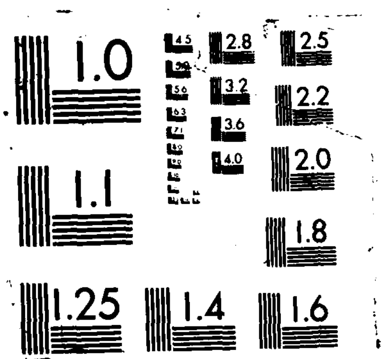
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THE SINGLETON RELATIVE CHEBYSHEV CENTER IN BEST SIMULTANEOUS APPROXIMATIONS

APPROVED:

E. W. Cheney

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THE SINGLETON RELATIVE CHEBYSHEV CENTER IN BEST SIMULTANEOUS APPROXIMATIONS

BY

TIMOTHY BRUCE KILLAM, B.S., B.A.

REPORT

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF ARTS

THE UNIVERSITY OF TEXAS AT AUSTIN

December, 1986

To my wife Jenny, who provides the inspiration necessary for me to succeed.

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This report began with a series of lectures in approximation theory given by Dr. E. W. Cheney during the period February 10 through March 17, 1986. My interest in the subject of best approximation grew throughout these discussions. I thank Dr. Cheney for his patience and guidance through the writing of this report. I also wish to thank my wife Jenny for the immeasurable amount of support and encouragement she has provided throughout this project.

Timothy Bruce Killam

The University of Texas at Austin
December, 1986

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Chapter 1

Introduction

This report will conclude an extensive study of the relative Chebyshev center and best simultaneous approximation theory. ~~We will~~ (The author considers the current concept of the Chebyshev center as developed by Garkavi, and ~~will ultimately~~ examines in depth the relative Chebyshev center and concentrates on the special circumstances when the relative Chebyshev center reduces to a single point (singleton set).

While the notations and definitions of the relative Chebyshev center and the best simultaneous approximation of elements are different, the ultimate aim and the general concept are the same. Since the field is relatively new, there doesn't appear to be any standardization among the contemporary literature concerning terminology or notation. Thus the relative Chebyshev center, the restricted Chebyshev center and the best simultaneous approximation are effectively equivalent. In fact, when the relative Chebyshev center is shown to be a singleton, we have the unique element of best simultaneous approximation.

The ~~Our~~ intent is to give a sampling of the major work that has been done in this field so far. But first, let us describe in a more elementary way, the basic idea behind the relative Chebyshev center.

1.1 The Best Approximation

One of the principal problems in approximation theory can be stated in the following general setting. Given a normed linear space X, we wish to approximate its elements. Now if U is a linear subspace in X, then we can use its elements as approximants. If $x \in X$ and $u \in U$, then we can interpret u as an approximation to x and quantify this approximation by $\|x-u\|$.

The obvious case of importance is the best approximation, which arises for a fixed x, when we want $\|x-u\|$ as small as possible. Now using the definition of distance from x to U as:

$$dist(x,U)=\inf_{u \in U}\|x-u\|,$$

we say that u is a best approximation to x if

$$\|x-u\|=dist(x,U).$$

The principal applications of this notion occur in the approximation of functions. For various choices of X and U, we must be concerned with the existence of best approximations, their unicity, their characterization and computation.

1.2 Best Simultaneous Approximation

Simultaneous approximation is concerned with the best approximation of sets of functions or elements, rather than just single ones. The most highly studied area so far is the problem of approximating two functions simultaneously.

Given the same setting as before, we now add a subset K of X. We desire to approximate all elements of K simultaneously by a single element of U. For $u \in U$, this is quantified by $\sup_{x \in K} \|x-u\|$. Then the best simultaneous approximation is an element u for which the expression is smallest, i.e.

$$\inf_{u \in U} \sup_{x \in K} \|x-u\|$$

This quantity has a name; it is called the **Chebyshev radius** of K relative to U and denoted by r(U;K). Then we can define the following:

$$Z(U;K)=\{u \in U \ ; \ \sup_{x \in K} \|x-u\|=r(U;K)\}$$

This is called the **Chebyshev center of K relative to U**, or the relative Chebyshev center of K in U.

Another way to consider this is to denote, for all $x \in X$, the closed sphere of radius r and center x by:

$$S(x,r)=\{y \in X \ ; \ \|x-y\| \leq r\}.$$

Then set

$$r(x,K)=\inf\{r \ ; \ K \subset S(x,r)\}$$

This leads to the relative Chebyshev radius:

$$r(U;K)=\inf\{r(x,K) \ ; \ x \in U\}$$

Then as before the set of relative Chebyshev centers of K in U is:

$$Z(U;K)=\{x \in U \ ; \ r(x,K)=r(U;K)\}$$

1.3 Special Cases

The above motivates many questions. For example, how do we know a best simultaneous approximation exists: that is, $Z(U;K) \neq \emptyset$? What characterizes the elements of $Z(U;K)$ and how can we determine them?

These questions have been examined by various authors in special settings. By placing restrictions on the three spaces involved above, we obtain some interesting special cases in which the structure of the spaces makes these questions (and others) easier to answer.

As stated, our particular interest is in the cases when the relative Chebyshev center reduces to a singleton. Thus it is important for us to examine these various special cases to see when we can be assured that we have a singleton, and hence a unique best simultaneous approximation.

In the following chapters we will show what has been accomplished in some special cases and give a few remarks on what areas are unexplored in the field. This report will conclude with an extensive bibliography of the major work in the field.

Chapter 2

Conditions on the Normed Linear Space

2.1 Strict Convexity

Let us begin by discussing the setting for the relative Chebyshev center, which, according to the definition, is a normed linear space X. Many authors [2, 4, 5, 8, 23, 27] relate the structural properties of relative centers to the convexity properties of the spaces. The major work here is that of Amir and Ziegler [5, 8] and deals with the case where X is strictly convex relative to U.

> **Definition 1**: The normed linear space X is said to be *strictly convex with respect to* a linear subspace U if the unit sphere of X contains no segment parallel to U., i.e. the conditions
>
> $$\|x\|=\|y\|=\left\|\frac{x+y}{2}\right\|,\quad x-y\in U$$
>
> imply that $x=y$

If X is strictly convex with respect to U then the relative Chebyshev center of a compact set $K\subset X$ in U is at most a singleton. To see this, let us assume that the radius, r(U,K), is 1 for simplicity. Now if $a,b\in Z(U;K)$, we can say that $\frac{a-b}{2}\in Z(U;K)$. This is true since we know

$$\sup_{x\in K}\|x-a\|=\sup_{x\in K}\|x-b\|=1$$

which implies that

$$\|x-a\|\leq 1 \ and \ \|x-b\|\leq 1$$

for all $x\in K$. So by the triangle inequality $\|x-\frac{a+b}{2}\|\leq 1$ for all $x\in K$. Therefore,

$$\sup_{x\in K}\|x-\frac{a+b}{2}\|\leq 1$$

and hence $\frac{a+b}{2}\in Z(U;K)$. Since K is compact, we know that there exists an $x\in K$ such that $\|x-\frac{a+b}{2}\|=1$. Since $r(U,K)=1$, we have $\|x-a\|=\|x-b\|=1$. Obviously, $(x-a)-(x-b)=b-a\in U$; the strict convexity of X then implies that $a=b$. Thus the center Z(U;K) is at most a singleton.

This is most important in the theory of best simultaneous approximation. For example, consider the case where K={x,y} in the above argument; then Z(U;x,y) is at most a singleton. This is the problem of best simultaneous approximation of two elements.

This problem is investigated more fully in [5, 6, 8] in an attempt to produce an element of the center. Amir and Ziegler demonstrate the following:

Lemma 2: Let X be a normed linear space, let $x,y\in X$, and let U a subspace of X. Suppose that $u\in Z(U;x,y)$. Then exactly one of the following alternatives hold:

- $\|u-x\|=\|u-y\|$

- $u \in Z(U;x)$ and $\|u-x\| > \|u-y\|$
- $u \in Z(U;y)$ and $\|u-y\| > \|u-x\|$

This lemma suggests a procedure for producing an element of Z(U;x,y). Compute first P(U;x) and P(U;y). Recall that Z(U;x) = P(U;x) is the set of best approximations from U to x (i.e. the metric projection of x into U; cf. [10]). If one of them is in Z(U;x,y) we are done. Otherwise, consider the line segment

$$[P(U;x),P(U;y)]=\{u \ ; \ u=\alpha P(U;x)+(1-\alpha)P(U;y), \ 0 \leq \alpha \leq 1\}$$

Choose an α such that $\|u-x\|=\|u-y\|$. If this is in Z(U;x,y), we are through. If not, the problem is reduced to a search in the set $\{v \in U \ ; \ \|v-x\|=\|v-y\|\}$.

However, suppose we already have an element of Z(U;x) and one of Z(U;y), then we can find an element of Z(U;x,y), provided that

$$Z(U;x,y) \cap [u,v] \neq \emptyset$$

for all $u \in Z(U;x)$ and $v \in Z(U;y)$.

Now if U is a subspace of a strictly convex normed linear space X, then basic approximation theory indicates that there is at most one best approximation to any element $x \in X$. Goel et al. [23] prove a similiar result for the best simultaneous approximation.

> **Proposition 3**: Let U be a subspace of a strictly convex normed linear space X. Then there is at most one best simultaneous approximation from the elements of U, to any pair of elements $x,y \in X$.

First note that an element $u \in U$ is said to be a best simultaneous approximation to x and y if:

$$dist(x,y:u)=\max\{\|x-u\|,\|y-u\|\}$$

Now consider the following proof of the proposition.

Proof: Suppose u_1 and u_2 are best simultaneous approximations to $\{x,y\}$. Let $d=\max\{\|x-u_i\|,\|y-u_i\|\}$, $i=1,2$. Then there are two cases to consider.

1. Let $\|x-u_1\|=d$ and $\|y-u_1\|=l<d$ (or vice-versa), and write $d-l=\epsilon$. We can find a convex neighborhood $A \subset U$ of u_1 such that:

$$d-\frac{\epsilon}{4} \leq \|x-u\| \leq d+\frac{\epsilon}{4}$$

and

$$l-\frac{\epsilon}{4} \leq \|y-u\| \leq l+\frac{\epsilon}{4}$$

for all $u \in A$. Thus $\max\{\|x-u\|,\|y-u\|\}=\|x-u\|$ whenever $u \in A$. Further $\|x-u\| \geq d$. The element

$$k=\lambda u_2+(1-\lambda)u_1 \ \in A$$

provided λ is sufficiently small and non-zero. Since k is also a best simultaneous approximation, in [23] we have $\|x-k\|=d$. However $\|x-u_1\|=d$ and $\|x-\frac{u_1-k}{2}\|=d$. From these last three relations and the strict convexity of the norm we deduce $u_1=k$, thus $u_1=u_2$.

2. Assume $\|x-u_1\|=\|y-u_2\|=d$, and also $\|x-u_2\|=\|y-u_1\|=d$ (if not, the previous argument holds) write $k=\frac{u_1-u_2}{2}$. Then there are three cases:

- $\|x-k\| = \|y-k\| = d$
- $\|x-k\| = d$ and $\|y-k\| < d$ or
- $\|x-k\| < d$ and $\|y-k\| = d$

In all the three cases we have either:

- $\|x-u_1\| = \|x-u_2\| = \|x-\frac{u_1-u_2}{2}\|$ or
- $\|y-u_1\| = \|y-u_2\| = \|y-\frac{u_1+u_2}{2}\|$

or both. Using the strict convexity of the norm we conclude $u_1=u_2$. Q.E.D.

Here again we have shown the uniqueness of the best simultaneous approximation in the setting of the strictly convex normed linear space X.

2.2 Uniform Convexity

Another approach to the relation between convexity properties and the nature of the center taken by many authors [5] is via the concept of uniform convexity.

Definition 4:

1. The space X is said to be *uniformly convex with respect to every direction in U* if for every z, $0 \neq z \in U$ and every $\epsilon > 0$, there exists a $\delta=\delta(z,\epsilon) > 0$ such that

$$\|x\| = \|y\| = 1,\ x-y=\lambda z,\ \left\|\frac{x+y}{2}\right\| > 1-\delta$$

implies that $\lambda < \epsilon$

2. The space X is *uniformly convex with respect to U* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \ x-y \in U, \ \|x-y\| > \epsilon$$

implies that $\|\frac{x-y}{2}\| > 1-\delta$.

Note that if X is uniformly convex with respect to every direction in U, then it is strictly convex with respect to U as in the previous section. Garkavi [20] showed that X is uniformly convex with respect to every direction in U if and only if Z(U;K) is at most a singleton for every bounded $K \subset X$. By generalizing Garkavi's result we have in [5]:

Theorem 5: The space X is uniformly convex with respect to every direction in U if and only if Z(U;K) is at most a singleton for every bounded $K \subset X$.

Proof:

1. Assume Z(U;K) is not a singleton, and let y_1, y_2 be two distinct elements of Z(U;K). Then $y_0 = \frac{y_1+y_2}{2}$ is also in Z(U;K). Select now a sequence $(x_n) \subset K$ such that

$$\|y_0 - x_n\| \to r(U;K).$$

Then

$$\|y_i - x_n\| \to r(U;K) \ \textit{for} \ i=1,2.$$

We may assume that $\|y_1 - x_n\| \geq \|y_2 - x_n\|$ and take $z_n = y_1 + t_n(y_2 - y_1)$ with $t_n \geq 1$ chosen so that $\|z_n - x_n\| = \|y_1 - x_n\|$. Then $u_n = \frac{y_1 - x_n}{\|y_1 - x_n\|}$ and $v_n = \frac{z_n - x_n}{\|z_n - x_n\|}$

satisfy $\|u_n+v_n\| \to 2$ while $u_n-v_n \in U$ and it does not tend to 0, so that the uniform convexity with respect to every direction in U condition is not satisfied.

2. Conversely, assume X is not uniformly convex with respect to every dirction of U. Then there exists an element $z \in U$ and two sequences $(x_n),(y_n)$ such that

   - $\|x\|=\|y\|=1$
   - $x_n-y_n=\lambda_n z$
   - $|\lambda_n| \geq \lambda > 0$
   - $\|\frac{x_n+y_n}{2}\| \to 1.$

   Let $u_n=\frac{x_n+y_n}{2}$, $K=\{\pm u_n \ ; \ n=1,2,...\}$. Since $\|u_n\| \to 1$, it follows that r(U;K)=1 and $0 \in Z(U;K)$. However, we have also $\pm\frac{\lambda z}{2} \in Z(U;K)$ since

   $$\|u_n \pm \frac{\lambda z}{2}\|=\|\{\frac{1}{2}\pm\frac{1}{2\lambda_n}\}x_n+\{\frac{1}{2}\pm\frac{1}{2\lambda_n}\}y_n\| < 1$$

   Hence, Z(U;K) is not a singleton. Q.E.D.

Thus we have a characterization for the singleton relative Chebyshev center when the normed linear space X is uniformly convex with respect to every direction of U. This is not surprising since the concepts of strict and uniform convexity are so close.

Chapter 3

Conditions on the Subspace

In this chapter we will consider a case previously mentioned; that is, when K is a set of functions to be approximated by an approximating family of functions U. This problem has been studied recently under various norms and various definitions of X. [5, 6, 8, 13, 23].

3.1 The Haar subspace

Let us consider the case when the linear subspace U is n-unisolvent. This appears to be a natural framework for expecting a unique best approximant. If we take U as an n-dimensional n-unisolvent linear subspace, then it would be called a **Haar subspace**.

Now [16] makes the simple observation that it is possible to reduce this to the case of two functions, the upper and lower envelopes. So consider X=C[0,1] as a simple example. If $K \subset X$ is compact, then the functions,

$$K_U(t)=sup\{f(t)\ ;\ f\in K\}$$
$$K_L(t)=inf\{f(t)\ :\ f\in K\}$$

are continuous. Furthermore, if $g \in$ C[0,1] we have

$$\begin{aligned} r(g,K) &= \sup\{\|f-g\| \ ; \ f\in K\} \\ &= \sup\{|f(t)-g(t)| \ ; \ f\in K, \ t\in[0,1]\} \\ &= \sup\{\max(K_U(t)-g(t), \ g(t)-K_L(t) \ ; \ t\in[0,1])\} \\ &= \max(\|K_U-g\|,\|g-K_L\|) \\ &= r(g;K_U,K_L) \end{aligned}$$

Hence, we can look at the problem of relative centers for pairs of functions (f,g) with $f\geq g$.

Suppose we take $U\subset X$ to be an n-parameter approximating family and define the relative Chebyshev center of (f,g) with respect to U by

$$Z(U;f,g)=\{u^*\in U \ ; \ r(u^*;f,g)=min(r(u;f,g) \ ; \ u\in U)\}$$

As pointed out in [8], u^* exists because of compactness. To proceed further, consider the following from [16]. The approximating families used below are extended n-unisolvent (non-linear) families, which include our case of interest: the Haar subspace. However, the following definitions are more general than the linear case for Chebyshev systems.

Definition 1: The n-parameter approximating family $U=\{u(a^*;t) \ ; \ a^*\in S\subset\mathbf{R}^n\}$ of functions defined on $[0,1]$ is *n-unisolvent* if for any given set $\{t_i\}^n{}_{i=1}$ of distinct points in $[0,1]$ and any set $\{y_i\}^n{}_{i=1}$ of arbitrary numbers there exists a unique a^* such that

$$u(a^*;t_i)=y_i \ \ for \ \ i=1,\ldots,n$$

Also, we need some ideas from the foundations of approximation theory. In [31] we have the following:

Definition 2: Let u,f,g be given, $f\geq g$. Then $\{t_i\}^k{}_{i=1}$ is a *$(u;f,g)$-alternance* if

$$f(t_{2i})-u(t_{2i})=u(t_{2j-1})-g(t_{2j-1})=r(u;f,g)$$

for all i and j, or

$$u(t_{2i})-g(t_{2i})=f(t_{2j+1})-u(t_{2j+1})=r(u;f,g)$$

for all i and j. Each point where $f(t)-u(t)=r(u;f,g)$ is called a (+)-point, and a point where $u(t)-g(t)=r(u;f,g)$ is called a (-)-point. Both kinds are called (e)-points.

Once again [16] gives us an important definition.

Definition 3: The point t_o is called a *straddle point* with respect to the $(u;f,g)$-approximation if it is both a (+)-point and a (-)-point, i.e. if

$$f(t_o)-u(t_o)=u(t_o)-g(t_o)=r(u;f,g).$$

Which leads us to an important theorem for the relative centers of (f,g) with respect to unisolvent families.

Theorem 4: Let $f,g \in C[0,1]$, with $f \geq g$ and let U be an n-unisolvent family in C[0,1]. Then $u \in Z(U;f,g)$ if and only if, either:

1. $(u;f,g)$ has a straddle point
2. $(u;f,g)$ has an n+1 alternance

Proofs of this theorem are available in [16] and also from Amir and Ziegler in [8] where they further state that if there exist n straddle points, then Z(U;f,g) is a singleton.

This characterizes the center, but doesn't really tell us much of uniqueness for it is not necessary that we have n straddle points for Z(U;f,g) to be a singleton. By careful examination of the situation for n

Hermite data, [6, 8] give us a more complete theorem using the extended n-unisolvent family.

Definition 5: The family $U=\{u(a^*;t)\ ;\ a^* \in S \subset \mathbf{R}^n\}$ is called an *extended n-unisolvent family* if for each set of n Hermite data

$$t_i, i=1,\ldots k;\quad y_i^j, i=1,\ldots k;\ j=0,\ldots n_{i-1};\ \sum_{i=1}^{k} n_i = n$$

there exists a unique a^* such that

$$u^{(j)}(a^*;t_i) = y^j_i,\ \ i=1,\ldots k;\ j=0,1,\ldots n_{i-1}$$

Thus, with the following definitions from [6], we have a full characterization of uniqueness.

Definition 6: A straddle point t_o is of *deficiency k* if k Hermite type data are imposed on the extremal functions at t_o.

Definition 7: Let $u \in Z(U;f,g)$ and let t be a straddle point, of deficiency k, which is a cluster point of (+)-points. Let m be the largest integer, $0 \leq m \leq n-k$, such that

$$u^{(k)}(t) = g^{(j)}(t),\ j=1,\ldots,k+m-1,$$

then $h=k+m$ is called the *total deficiency* of t.

Theorem 8: Let $f,g \in C^{(n)}(I)$, $f \geq g$, and let U be an extended n-unisolvent family. Then $Z(U;f,g)$ is a singleton if and only if either

1. $\sum_{i=1}^{k} h_i \geq n$

   where $h_1,\ldots h_k$ are the total deficiencies of the straddle points

2. there exists a function $u^* \in Z(U;f,g)$ such that $S^+(\alpha) \geq n$

where $S^-(\alpha)=\sup\{S^+(\alpha(t_1),\ldots,\alpha(t_k))\}$ and $S^+(x)$ is the maximal number of sign changes of the components of vector x. So α is the function corresponding to u^*.

It should be noted that the characterization of a singleton relative Chebyshev center in the above theorem does not depend on the linearity of U. However, since we are demonstrating the simplification of the general relative center to the the problem of just two functions: (f,g) with $f \geq g$, the restriction to the Haar subspace allows us to take advantage of the above.

Chapter 4

Existence of Centers

4.1 The Chebyshev Center

The concept of the Chebyshev center was introduced in 1964 by Garkavi [20] and most of the basic results are due to him. Concerning the problem of existence, he observed the following general existence principle of Chebyshev centers for bounded subsets of a closed subspace V in Banach space X.

> **Proposition 1**: If V carries another topology τ such that $\tau(y_\alpha)=y$ implies $dist(x,y) \leq \liminf dist(x,y_\alpha)$ for all $x \in X$, then a τ-accumulation point of a *minimizing sequence* for the Chebyshev radius is necessarily a Chebyshev center. In particular, if X is a dual normed space, then for every w^*-closed subspace $V \subset X$, $Z_V(A)$ is nonempty for all A in the collection of bounded sets of X.

This is in the classical $L_p(\mu)$ spaces, where $1 \leq p < \infty$. The existence of Chebyshev centers for bounded sets in C(T) where T is a compact Hausdorff space, was established by Kadets and Zamyatin [24].

In a more general situation, consider T as an arbitrary topological space. Then C(T) is the Banach space of all bounded real-valued, continuous functions defined on T. In this setting, Franchetti and Cheney [17] provide us with a clearer theorem.

Theorem 2: Let T be any topological space, and let K be any subset of C(T). Then the Chebyshev center of K is nonempty.

The proof of this theorem is in reality the special case of a compact topological space T′, and is shown in [37].

4.2 The Relative Chebyshev Center

The above is concerned with the unrestricted center. That is U=X=C(T). But suppose we want to take U as a subspace of C(T). Here existence is characterized by the following in [17].

Theorem 3: Let K be a compact set and U a subspace of C(T), T arbitrary. Then a point u_0 of U belongs to the relative Chebyshev center Z(U;K) if and only if either:

1. $c+w-\|w\| \leq u_0 \leq c-w+\|w\|$, *or*
2. there is an $f \in U^{\perp}$ such that $\|f\|=1$ and

   $$|f(c)+|f|(w)| = \|w+|c-u_0|\|$$

   here c and w are defined by

   $$c = \frac{1}{2}(K_U+K_L)$$

   and

   $$w = \frac{1}{2}(K_U-K_L)$$

   where K_U and K_L are as defined in Chapter 3.

Chapter 5

Final Remarks

In the preceeding chapters we have considered the relative Chebyshev center and examined the cases when it can be shown to be a singleton set. Since the singleton center is of great importance for the best simultaneous approximation, it seems that one goal should be a complete characterization of its existence. However, there is still much work to be done in this area.

Chapter 2 demonstrated the existence of the singleton relative Chebyshev center when X is strictly convex with respect to a subspace U. Unfortunately, some very practical situations do not fit into this framework. Amir and Ziegler [6, 8] point out the following:

> **Proposition 1**: Let μ be any measure. Then $L_1(\mu)$ is not strictly convex with respect to U if $dim U \geq 2$. If μ is *nonatomic*, then $L_1(\mu)$ is not strictly convex with respect to any subspace.

If μ is nonatomic, then this statement is simply a consequence of the fact that $L_1(\mu)$ has no finite-dimensional Chebyshev subspaces [31].

> **Proposition 2**: The space $C_0(T)$ is not strictly convex with respect to any subspace U with $dim U \geq 2$.

Proposition 3: The space $(C[a,b],\|\cdot\|_1)$ of continuous functions with the L_1-norm is not strictly convex with respect to U if $dim U \geq 2$.

These results show that the strict convexity hypotheses are somewhat restrictive. So we can only be assured of the existence of a singleton relative Chebyshev center for the first dimension in some cases. It appears the other theories presented here are just as restrictive.

Chapter 3 limits us to a Haar subspace for U, which is finite-dimensional by definition. While the general theory developed for n-unisolvent approximating families is non-linear, we restrict the general case to the Haar subspace so that we can characterize the singleton relative Chebyshev center for the limited case of two functions, the upper and lower envelopes. By attempting to define the unique element of best simultaneous approximation, we find ourself restricted to an examination of the total deficiencies of the straddle points which are based on cluster points in the alternance. While the theory here is promising, it too is far from complete.

Characterizing the singleton relative Chebyshev center should be viewed more as a goal in simultaneous approximation than as something realistic and easily attainable. As with most problems, the higher the dimension the more difficult it is to characterize the circumstances under which the singleton center occurs, if it does at all.

While Chapter 4 made only a brief examination of the subject, knowing when the relative Chebyshev center must exist is of utmost importance in determining when it might reduce to a singleton. The

theory here too is restrictive. For we have only a characterization in the Banach spaces of all bounded real-valued continuous functions on a topological space.

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